The Vector Potential

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1 Introducing A

In electrostatics we found that the fact that $\nabla \times \mathbf{E} = 0$ allowed us to define a scalar potential with the property that $\nabla V = -\mathbf{E}$. This electric potential gave us ways of visualising particles moving in electric fields in terms of climbing the 'potential hill', increasing or decreasing their electric energy. We also saw it gave us a fully general way of calculating electric fields without referring to any symmetry or dealing with any vectors in terms of Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \tag{1}$$

which has the solution

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'. \tag{2}$$

So if we know the charge distribution[1] ρ we have a straight forward, algorithmic way of finding the electric field without appealing to any symmetry or dealing with vector addition. For simple distributions this can be calculated by hand or for complicated distributions can be done by a computer (which you did in Oblig 1).

It seems reasonable to ask weather we can do exactly the same thing for magnetic fields, but this can not be since magnetic fields are not irrotational. Ampere's law states that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{3}$$

so we can not have scalar potential defining $\mathbf{B}$ such that $\nabla V_b = -\mathbf{B}$ when the curl of any gradient field is zero. This would be a contradiction to Ampere's law! However as a consequence of there not being any magnetic monopoles we know that magnetic fields are solenoid fields[2], meaning that they are divergence less, which is encapsulated in $\nabla \cdot \mathbf{B} = 0$. This fact allows us to define a *vector potential* $\mathbf{A}$ with the property that $\mathbf{B} = \nabla \times \mathbf{A}$. We want a magnetic equivalent of equation 1 so let's try to relate $\mathbf{A}$ to the current density $\mathbf{J}$. Using Ampere's law (equation 3) we get

$$\begin{aligned}
\nabla \times \mathbf{B} &= \nabla \times (\nabla \times \mathbf{A}) && (4)\\
&= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} && (5)\\
&= \mu_0 \mathbf{J} && (6)
\end{aligned}$$

[1] Often this is the case since we in a given experiment can distribute the charge as *we* like.

[2] The magnetic field inside a solenoid is constant and therefore divergence less. This might be the reason for calling divergence less fields for solenoid fields.

where we have used an identity for the curl of the curl of a vector field.[3] Thus our relation reads

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A} = \mu_0\mathbf{J}. \tag{7}$$

Looking back at Poisson's equation (equation 1) this looks a bit more nasty. If it we're not for the pesky $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})$ term at least our equation would look like vector form of Poisson's equation. This term include $\boldsymbol{\nabla}\cdot\mathbf{A}$, but what is the divergence of $\mathbf{A}$? The answer is that it is up *us* to define it! Remember that the electric potential V was only defined up to a constant. If $\boldsymbol{\nabla}V = -\mathbf{E}$ we could define a new potential $V' = V + c$ and this would give us exactly the same electric field since

$$\boldsymbol{\nabla}V' = \boldsymbol{\nabla}(V + c) = \boldsymbol{\nabla}V + 0 = -\mathbf{E}$$

meaning that we are not really changing the physics by adding a constant to V. Similarly we can add any gradient field $\boldsymbol{\nabla}\Lambda$ to $\mathbf{A}$ since the curl of any gradient field is zero and therefore this term would vanish in taking the curl of $\mathbf{A}$. Thus we can define

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Lambda$$

and we'll still have

$$\boldsymbol{\nabla}\times\mathbf{A}' = \boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\Lambda) = \boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$$

and this allows us to define $\boldsymbol{\nabla}\cdot\mathbf{A}$ since

$$\boldsymbol{\nabla}\cdot\mathbf{A}' = \boldsymbol{\nabla}\cdot A + \boldsymbol{\nabla}^2\Lambda.$$

So if we want to clean up our relation in equation 7 we just choose Λ such that the divergence of $\mathbf{A}$ is zero. This leads us to the result

$$\boxed{\boldsymbol{\nabla}^2\mathbf{A} = -\mu_0\mathbf{J}.} \tag{8}$$

which indeed looks very much like Poisson's equation. The only difference is really that we now have one equation for each component of $\mathbf{A}$, but we reached our goal in finding an equation which we can solve algorithmically.

2 Calculating the vector potential

We set out to find the magnetic analog of Poisson's equation and ended up with equation 8. We can exploit the fact that this equation and equation 1 are similar. Looking just at the x-component of equation 8 we have

$$\boldsymbol{\nabla}^2 A_x = -\mu_0 J_x.$$

which has the exact mathematical form of Poisson's equation. We know that the solution to

$$\boldsymbol{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}$$

is

[3]This identity will also show up in the derivation of electromagnetic waves, so it might be worthwhile to get comfortable with it now.

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'.$$

and identifying $\mu_0 J_x$ with ρ/ε_0 we can conclude that

$$A_x = \frac{\mu_0}{4\pi}\int \frac{J_x(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'$$

or more generally

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'}. \tag{9}$$

With this equation it is in principle straight forward to calculate the vector potential given a distribution of currents. Then it's just to take the curl to find the magnetic field.

3 The vector potential in electrodynamics

$\mathbf{A}$ and V are usefull in electrostatics because they give us a straight forward way of calculating $\mathbf{B}$ and $\mathbf{E}$ without appealing to any symmetry and we can find it by solving tree Poisson's equations. However the scalar and vector potential becomes even more important when we enter the domain of electrodynamics. When we have moving charges and changing currents the laws of interaction between sources and fields are expressed in in terms of Maxwell's four equations

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{10}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{11}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{12}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}. \tag{13}$$

Now because a changing $\mathbf{B}$ causes a curl in $\mathbf{E}$, $\mathbf{E}$ will no longer be the gradient of any scalar V, but let's try to find a quantity that will. $\mathbf{B}$ is still divergence less so $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$ and substitution into equation 11 yields

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times A) = -\boldsymbol{\nabla}\times(\frac{\partial\mathbf{A}}{\partial t})$$

which implies that

$$\boldsymbol{\nabla}\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right) = 0$$

and thus the sum of the electric field and the time derivative of the vector potential will be an irrotational field so *it* can be expressed as the gradient of a scalar as

$$\boldsymbol{\nabla}V = -\mathbf{E} - \frac{\partial\mathbf{A}}{\partial t}. \tag{14}$$

When $\mathbf{A}$ does not change with time this just reduces to our old V from electrostatics and therefore is a valid generalization of V for electrodynamics. We want only the potentials in terms of the sources $\mathbf{J}$ and ρ and we can eliminate $\mathbf{E}$ by using Gauss law (equation 10) by taking the divergence of equation 14. This gives us

$$\boldsymbol{\nabla}^2 V = -\boldsymbol{\nabla}\cdot E - \boldsymbol{\nabla}\cdot\left(\frac{\partial \mathbf{A}}{\partial t}\right) = -\frac{\rho}{\varepsilon_0} - \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right)$$

such that

$$\boldsymbol{\nabla}^2 V + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right) = -\frac{\rho}{\varepsilon_0}.$$

This equation relates V and $\mathbf{A}$ to the charge distribution ρ, but we also need a relation in terms of the current distribution $\mathbf{J}$. This is obtained by Ampere's law (equation 13) by inserting the vector potential and our relation between V and $\mathbf{E}$ from equation 14 such that

$$\boldsymbol{\nabla}\times B = \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\mathbf{A}\right) = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}V - \frac{\partial \mathbf{A}}{\partial t}\right) = \mu_0\mathbf{J} - \mu_0\varepsilon_0\boldsymbol{\nabla}\left(\frac{\partial V}{\partial t}\right) - \mu_0\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}$$

using the identity for the curl of the curl and rearranging terms we get

$$\boldsymbol{\nabla}^2\mathbf{A} - \mu_0\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J} \tag{15}$$

and together with

$$\boldsymbol{\nabla}^2 V + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right) = -\frac{\rho}{\varepsilon_0}. \tag{16}$$

these two equations embody all the information in the four Maxwell equations!

4 Gauge transformations

Now you might argue that the two equations found in the previous section looked kind of hairy compared to the neat Maxwell equations and I would agree. Even though we did reduce the six problems of finding the components of $\mathbf{E}$ and $\mathbf{B}$ in to four (finding V and the components of $\mathbf{A}$) these equations seems very hard to solve due to their hairy nature. But remember what we did back in section 1 where we exploited the fact that adding a gradient field $\boldsymbol{\nabla}\Lambda$ to $\mathbf{A}$ isn't changing the physics because $\boldsymbol{\nabla}\times\mathbf{A}$ is still $\mathbf{B}$ and used this to clean up our equation by making $\mathbf{A}$ divergence less. The process changing V and $\mathbf{A}$ in such a way that $\mathbf{E}$ and $\mathbf{B}$ does not change is called a *gauge transformation* and we'll now use such transformations to clean up our dynamical equations 15 and 16. Now in electrostatics $\mathbf{E}$ was uniquely determined by V, but in electrodynamics $\mathbf{E}$ is also determined by $\mathbf{A}$ trough

$$\mathbf{E} = -\boldsymbol{\nabla}V - \frac{\partial\mathbf{A}}{\partial t}.$$

So if we're adding $\boldsymbol{\nabla}\Lambda$ to $\mathbf{A}$ we are not changing $\mathbf{B}$, but we're changing $\mathbf{E}$ to

$$\mathbf{E}' = -\boldsymbol{\nabla}V - \frac{\partial\mathbf{A}}{\partial t} - \boldsymbol{\nabla}\left(\frac{\partial\Lambda}{\partial t}\right) = -\boldsymbol{\nabla}\left(V + \frac{\partial\Lambda}{\partial t}\right) - \frac{\partial\mathbf{A}}{\partial t}.$$

This is not allowed for our transformation to be a gauge transformation and we therefore have to compensate by simultaneously subtracting $\partial\Lambda/\partial t$ from V. This ensures that neither $\mathbf{E}$ nor $\mathbf{B}$ changes. So to conclude a gauge transformation must be of the form

$$V' = V - \frac{\partial \Lambda}{\partial t} \qquad \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Lambda. \tag{17}$$

The transformation we did back in section 1 where we sat the divergence of $\mathbf{A}$ to zero by choosing

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Lambda$$

such that

$$\boldsymbol{\nabla}^2\Lambda = -\boldsymbol{\nabla} \cdot \mathbf{A}$$

is called the *Coulomb gauge* and is particularly usefull in electrostatics because of the simple relations we get for $\mathbf{A}$ and V (equations 8 and 1). In electrodynamics however, we wish to make the equations 15 and 16 as simple as possible and a good choice here is to set

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\mu_0\varepsilon_0 \frac{\partial V}{\partial t} \tag{18}$$

which gets rid of one of the terms in equation 15 and we obtain

$$\boxed{\boldsymbol{\nabla}^2\mathbf{A} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}} \tag{19}$$

and

$$\boxed{\boldsymbol{\nabla}^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}}. \tag{20}$$

In these two beautiful equations we have embodied everything in electrodynamics and we can in principle find everything that we could ever wish for. Notice that they reduce to our familiar static equations when $\mathbf{A}$ and V does not depend on time.