

# Note 1 – Essentials of vector calculus

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Electromagnetism is heavily based upon the mathematics of vector calculus. In general, vectors are multi dimensional, but in the areas of electromagnetism that we're going to cover, we will focus mainly on three dimensional vectors. These are living in the Euclidian space  $\mathbb{R}^3$ , but if you haven't heard about that space yet, you don't have to worry. It is simply the space made up by three coordinates,  $x$ ,  $y$  and  $z$ , that you know so well from before.

Even if we're only focusing on three (or sometimes two) dimensional vectors, there is a lot to know about how to work with these vectors, especially when we get into vector fields, where every point in space has a vector appointed to it. These are extremely useful to represent electric and magnetic fields that surrounds us everywhere. Understanding the properties of these fields is among the main goals in this course.

If you should feel the first section in this note is too easy, skip right ahead to section 1.4 about vector products. This will most likely be new to a lot of you, especially if you have not had the course in mechanics (MEK1100) earlier. Those of you who have had mechanics might find this whole note trivial, but you should still make sure you skim through it and are comfortable with all the topics. Also, when you're done with this note, you may dive deeper into divergence and flux in note 2 and 3.

## 1 Vector algebra

For starters, we'll have a quick look at vector algebra. Most of you are probably very familiar with this from before, but we still need to put a few simple definitions on the table to make sure we're all talking about the same thing and using the same notation. Or, at least, that you'll understand our notation throughout the course.

First of all, we'll represent vectors using bold face. In the lectures, an underline or overline might be used, since it's hard to draw boldface on a chalkboard. A simple vector might be like this:

$$\mathbf{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = 2.7\hat{\mathbf{i}} + 6.1\hat{\mathbf{j}} - 8.5\hat{\mathbf{k}}$$

All vectors start in the origin and end at their respective coordinates. In this case, the end point is located at  $x = 2.7$ ,  $y = 6.1$  and  $z = 8.5$ . We call the values of  $x, y$  and  $z$ , the *components* of the vector.

Adding and subtracting vectors is done simply by adding and subtracting each component. Let's take a quick example with the vector  $\mathbf{a} = 2\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$  and  $\mathbf{b} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}$ . The sum is

$$\mathbf{v}_{\text{sum}} = \mathbf{a} + \mathbf{b} = (2 + 4)\hat{\mathbf{i}} + (1 + 3)\hat{\mathbf{j}} + (4 + 1)\hat{\mathbf{k}} = 6\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$

And if we take the difference by subtracting  $\mathbf{b}$  from  $\mathbf{a}$ , we get

$$\mathbf{v}_{\text{diff}} = \mathbf{a} - \mathbf{b} = (2 - 4)\hat{\mathbf{i}} + (1 - 3)\hat{\mathbf{j}} + (4 - 1)\hat{\mathbf{k}} = -2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

It is as simple as that!

Furthermore, we need to make a statement about unit vectors. Unit vectors are the ones with length 1 and will be written with a hat ( $\hat{\phantom{x}}$ ) to distinguish them from all other vectors. Any vector  $\mathbf{v}$  can have

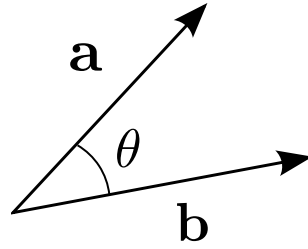


Figure 1: Two vectors  $a$  and  $b$  with an angle  $\theta$  between.

a unit vector  $\hat{\mathbf{v}}$  assigned to it. The unit vector would then be pointing in the same direction as  $\mathbf{v}$ , but have a length 1, no matter how long  $\mathbf{v}$  actually is.

Now, lets move on to the peculiarity of vector multiplication. There are two kinds of vector products, namely the *scalar product*  $\mathbf{a} \cdot \mathbf{b}$  and the *vector product*  $\mathbf{a} \times \mathbf{b}$ .

## 1.1 Scalar product

The scalar product is also known as the *dot* product, obviously because we use a dot to indicate the operation of the product. You might say that you're *dotting* two vectors. The name *scalar* product comes from the fact that what you end up with is a number, in opposition to the vector product where you end up with another vector. The scalar product is defined as

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta \quad (1)$$

where  $a$  and  $b$  are the magnitudes and  $\theta$  is the angle between the vectors if you place them tail-to-tail as shown in figure 1. While this definition is useful when you know the magnitude and direction of the vectors, there is another form of the definition which is useful when you know the elements of the vectors:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (2)$$

The scalar product is *commutative*, which means it does not matter in which order you multiply the two vectors:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \quad (3)$$

In addition, it is *distributive*:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \quad (4)$$

A great way to picture the scalar product is as the projection of  $\mathbf{a}$  along  $\mathbf{b}$ , times  $a$ . That is, how much of  $\mathbf{a}$  is pointing in the direction of  $\mathbf{b}$ , times the magnitude of  $\mathbf{a}$  (or vice versa).

The dot product of a vector with itself gives the magnitude, or length, squared,

$$\mathbf{a} \cdot \mathbf{a} = a^2 \quad (5)$$

## 1.2 Vector product

The vector product is also known as the *cross* product, and uses the cross,  $\times$ , to indicate its operation. You might say that you're *crossing* two vectors.

The definition of the vector product is represented on component form by calculating the following determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

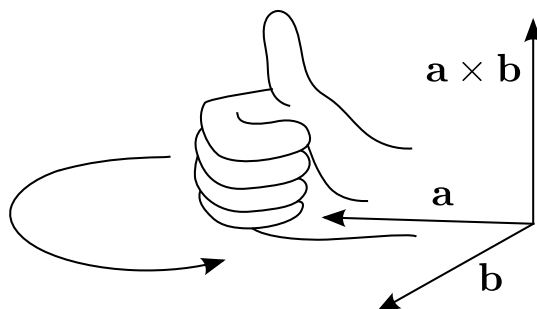


Figure 2: The right hand rule: Point all your fingers, except your thumb, in the direction of  $\mathbf{a}$ . Curl your fingers towards  $\mathbf{b}$ . If you pull up your thumb, it will be pointing in the direction of  $\mathbf{a} \times \mathbf{b}$ .

However, as with the dot product, we often know the magnitude of the vectors and the angle between them. In this case, another useful definition of the vector product is

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{n}}$$

where  $\hat{\mathbf{n}}$  is the unit vector pointing in the perpendicular direction to the plane spanned by the two vectors. Since there are two perpendicular directions to any plane, we use the right hand rule to choose which way to use.

The right hand rule is given by pointing your fingers in the direction of the first vector  $\mathbf{a}$  and curling your hand towards the second vector  $\mathbf{b}$ . If you stick your thumb out, it will now point in the direction of  $\mathbf{a} \times \mathbf{b}$  and  $\hat{\mathbf{n}}$ , as you see in figure 2.

This means that the vector product is *not commutative*, since doing the right hand rule in the other order (fingers at  $\mathbf{b}$  and curling towards  $\mathbf{a}$ ) would have your thumb pointing in the opposite direction. The length is still the same, it is just the direction which is inverted, so

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$$

But the vector product is still *distributive*,

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

### 1.3 Scaling a vector

Let us note quickly that you may also scale a vector. This is done by multiplying the vector  $\mathbf{v}$  with a number  $a$  and results in a vector in the same direction as  $\mathbf{v}$ , but  $a$  times the length:

$$a\mathbf{v} = a(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = ax\hat{\mathbf{i}} + ay\hat{\mathbf{j}} + az\hat{\mathbf{k}}$$

### 1.4 Triple products

The triple products are probably new to a lot of you, but are not too hard to grasp. In fact, they are more like simple rules that eases a few calculations drastically.

There are two triple products, since we can only obtain another vector by crossing two vectors. The resultant vector after a cross product can again be dotted or crossed by a third vector.

#### 1.4.1 Scalar triple product

The scalar triple product is given by

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

What's important to note about this product is that we can *shift* the order of the vectors, as long as the indexes are always appending,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$

The same is valid for descending indexes,

$$\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$$

Another useful property is that you may switch the dot and the cross,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

### 1.4.2 Vector triple product

The impressive thing about the vector triple product is that it lets you get rid of cross products altogether. If you perform a vector triple product with three arbitrary vectors, you get

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

## 2 Fields

In physics, fields may represent different physical quantities for all points in space. In this course we'll focus on the fields caused by electromagnetic elements. In the fields we'll be working with, every point in space is either represented by a scalar or a vector, so they are named *scalar fields* and *vector fields*.

Fields are basically functions, which return either a scalar or vector given any point in space. An example of a scalar field could be

$$f(x, y, z) = x^2 + 3y + 4z^9 + 2$$

where you simply input the coordinates and get a scalar value in return. In comparison, a vector field could be defined as

$$\mathbf{v}(x, y, z) = (z + 2y)\hat{\mathbf{i}} + (3y - 2z)\hat{\mathbf{j}} - 2x^3\hat{\mathbf{k}}$$

You should note that the  $x$ -component of the returned vector may be completely independent of the  $x$  value in the point, as in this example. The same of course applies for the two other components.

### 2.1 Scalar field

The most central scalar field you'll meet this semester is the electric potential. In addition, we'll have a look at the magnetic potential. We won't go deep into the properties of these potentials right now, but one important thing should be noted; the electric potential is not the same as the potential energy. They are very much related, but for now you should remember that they are not the same.

With that said, the scalar field associates a number to every point in space. In electromagnetism you'll be able to ask what the potential is at a specific point  $(x, y, z)$  and receive an answer telling you it is  $x$  volts. Equivalently, you might ask what the temperature is at any point in the room you are in, and receive a numerical answer.

#### 2.1.1 Equipotential

While we're already studying scalar fields, let us also mention the equipotential. These are the areas in space where the potential is equal. This is very useful when speaking of potential energy, since you can move along an equipotential without changing the energy.

In almost any scalar field, you will be able to draw in the equipotential. In figure 3, you can see the equipotentials around the charges in a dipole.

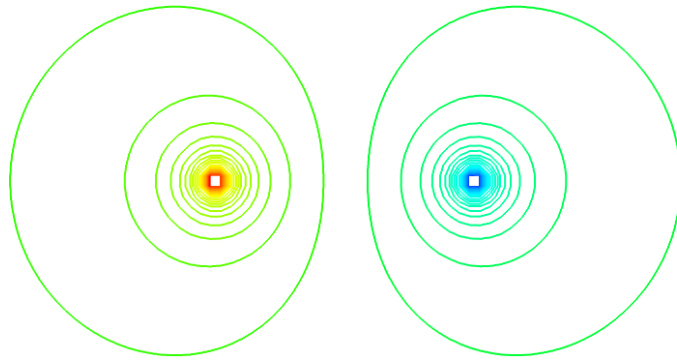


Figure 3: The lines show the equipotentials around the two charges in a dipole.

## 2.2 Vector field

In comparison, a vector field associates a vector to every space point. This might be a bit hard visualize mentally, but if you only look at a few points in space and associate a vector arrow to each point, you have a strong tool for visualizing the strength and direction of for instance an electric or magnetic field. The general form of a vector field is

$$\mathbf{V}(\mathbf{r}) = V_x(\mathbf{r})\hat{\mathbf{i}} + V_y(\mathbf{r})\hat{\mathbf{j}} + V_z(\mathbf{r})\hat{\mathbf{k}}$$

where  $\mathbf{r}$  denotes a point in space.

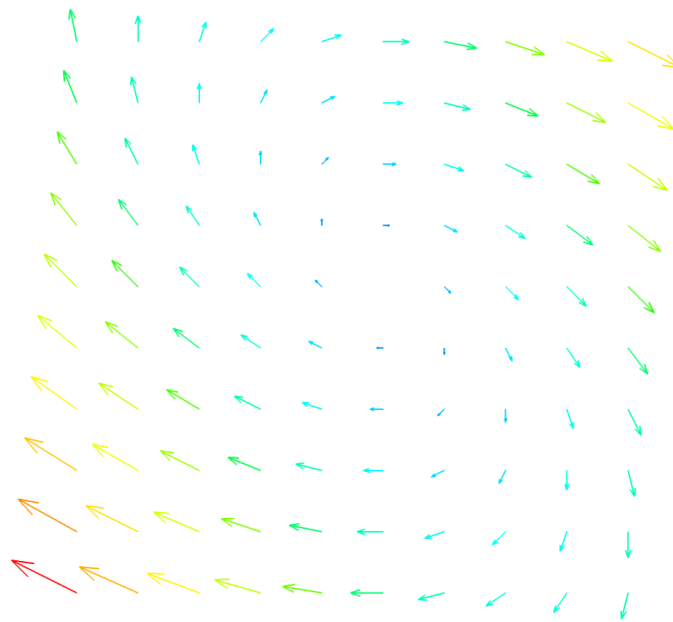


Figure 4: A vector field. All arrows represent the field at the point they are starting in. The length and color of the arrows indicate the strength of the field in every point. The direction of the arrow indicates the direction of the field.

You'll later learn that you may represent an electric field either with vectors or by it's corresponding electric potential, with their relation given by

$$\mathbf{E} = -\nabla V$$

Now, this equation might look simple, but it is actually hiding some complexity many of you are not yet familiar with; namely the del operator,  $\nabla$ . Which leads us to the section on differential calculus.

### 3 Differential calculus

In this course you'll need some tools to differentiate and integrate the fields we're working with. The protagonist in the play of differential calculus is the del operator,  $\nabla$ , which plays the role of *gradient*, *divergence* and *curl* on a field. To detect what role it inhibits, you have to look at what you find in-between the  $\nabla$  and the following function or field. If  $\nabla$  is alone, as in  $\nabla f$ , it acts as *gradient*. If it is together with a dot,  $\nabla \cdot f$ , it is acting as *divergence*. And if it joins with a cross,  $\nabla \times f$ , it plays its final role of *curl*.

How does it do that? Well, it is a pseudo-vector. That means it is not a vector by definition, since it does not have an assigned magnitude or direction, but it acts like a vector. It is defined as

$$\nabla = \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}$$

The del operator may perform the vector products we mentioned on the first pages of this note, even though it is not a "real" vector.

#### 3.1 Gradient

You're most likely familiar with the gradient from courses such as MAT1110, but we'll take our time to review it here. When the gradient acts on a function,  $f(x, y, z)$ , it returns a vector pointing in the direction of the greatest rate of increase of the scalar field, and the magnitude of the vector is the rate of increase.

The gradient is given whenever the del operator works on a scalar function without any operator in-between,

$$\nabla f = \frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} + \frac{\partial f}{\partial z} \hat{\mathbf{k}}$$

#### 3.2 Divergence

The scalar product between the del operator and a vector field,  $\mathbf{V}(\mathbf{r})$  defines the divergence,

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

This is a very important and central topic in this course, so we have covered it in a note of its own. Look for the note on divergence on our webpages:

<http://elektromagnetisme.no/notes/>

#### 3.3 Curl

If we on the other hand perform a cross product between the del operator and a vector field, we end up with the curl,

$$\nabla \times \mathbf{V} = \left( \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \hat{\mathbf{i}} + \left( \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \hat{\mathbf{j}} + \left( \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \hat{\mathbf{k}}$$

which may also be represented as the determinant of a matrix,

$$\nabla \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

As with divergence, the curl is very important in this course. You find more information about it in the note on curl on our webpages:

<http://elektromagnetisme.no/notes/>