

Note 2 - Flux

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1 What is flux?

In physics *flux* is the measure of the flow of some quantity through a surface S . For example imagine the flow of air or water through a filter. The flow the fluid both have a direction and a magnitude at every point and is therefore best represented as a vector field \mathbf{F} . phys

If you held a filter down in a river, the flow of water through the filter would depend on the area of the filter, what angle you held the filter relative to the flow, as well as how much water the river carried per second (figure ??). The relevant quantities are then the area A , the flow \mathbf{F} and the angle θ . We define the flux through a planar surface as

$$\Phi = \mathbf{F} \cdot \hat{\mathbf{n}}A = \mathbf{F} \cdot \mathbf{A} = FA \cos \theta \quad (1)$$

where $\hat{\mathbf{n}}$ is the *unit normal* vector to the surface and $\cos \theta$ conveniently shrinks the product when the plane is slanted by an angle θ . Furthermore we define the area vector \mathbf{A} as

$$\mathbf{A} = A\hat{\mathbf{n}} \quad (2)$$

such that the vector is normal to the surface and it's magnitude is the area of the surface.

Now this definition of flux assumes that the surface is a plane and that the flow \mathbf{F} is uniform. For nonuniform fields and more complicated surfaces we divide the surface S into smaller and smaller pieces ΔA and sum up the contributions from each piece (figure ??). In the limit where these pieces shrink to zero we generalize our definition and compute the flux as an integral

$$\Phi = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}dA = \int_S \mathbf{F} \cdot d\mathbf{A}. \quad (3)$$

Note that there is some ambiguity in the definition of the normal vector. It can point in both directions perpendicular to the surface. It's generally up to you to choose where you want your unit normal to

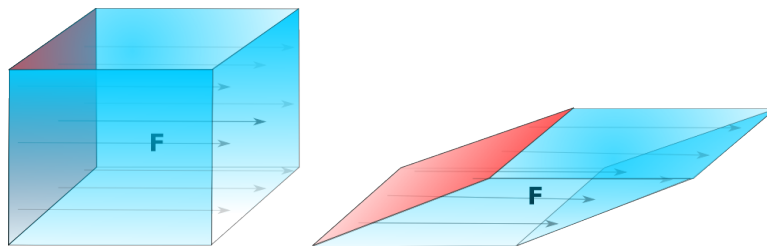


Figure 1: Flux through a surface depends on the angle of the surface relative to the flow.

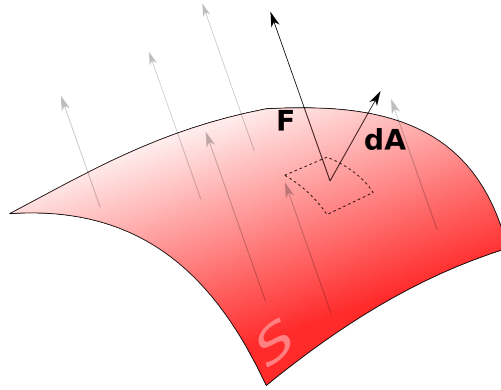


Figure 2: For non uniform fields and complicated surfaces we use integration.

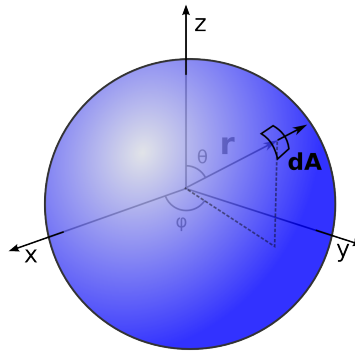


Figure 3: To construct a normal vector to the sphere we note that the radial vector \mathbf{r} has length a as long as it's pointing along the surface of the sphere.

point, the difference lies only in what you count as positive and negative flux, but when we have a *closed* surface we want to point our normal vectors out from the surface so that what flows out of that surface is counted positively. In the following we'll see how to compute these integrals for some relatively simple surfaces.

2 Flux through a sphere

If the surface we are considering is a sphere of radius a centered at the origin we know that the position vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is radial and therefore already normal to the surface of the sphere. To construct a unit normal to the sphere at a point (x, y, z) along the surface of the sphere, we divide \mathbf{r} by a , so that

$$\hat{\mathbf{n}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = \frac{\mathbf{r}}{a} = \hat{\mathbf{r}}.$$

Now an area element¹ along the sphere is best expressed in polar coordinates as $dA = a^2 \sin\theta d\phi d\theta$, such that

$$d\mathbf{A} = \hat{\mathbf{n}}dA = \frac{\mathbf{r}}{a}a^2 \sin\theta d\phi d\theta = \mathbf{r}a \sin\theta d\phi d\theta.$$

¹You might not have heard about an area element before, but it is nothing more than the Jacobi determinant from multivariable calculus.

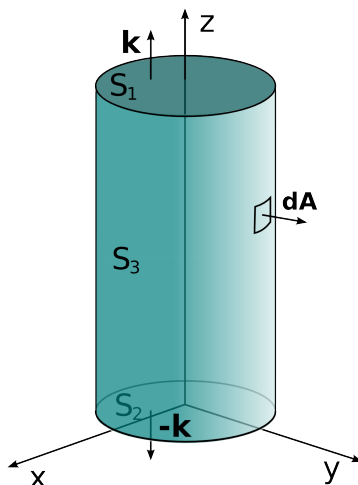


Figure 4: Flux through

2.1 Example

We want to find the flux of $\mathbf{F} = cz\hat{\mathbf{k}}$ through the surface \mathcal{S} of a sphere with radius a centered at the origin where c is a constant.

First we find

$$\mathbf{F} \cdot d\mathbf{A} = cz\hat{\mathbf{k}} \cdot \mathbf{r} a \sin \theta d\phi d\theta = caz^2 \sin \theta d\phi d\theta = ca^3 \cos^2 \theta \sin \theta d\phi d\theta$$

where $z = a \cos \theta$. The integral then becomes

$$\Phi = \int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{A} = \int_0^{2\pi} \int_0^\pi ca^3 \cos^2 \theta \sin \theta d\phi d\theta = \frac{4\pi a^3}{3} c.$$

3 Flux through a cylinder

For a cylinder with the axis of symmetry along the z -axis the surface \mathcal{S} is really constructed of three separate surfaces \mathcal{S}_1 , \mathcal{S}_2 and \mathcal{S}_3 as shown in figure ???. This means that we must compute

$$\Phi = \int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{A} = \int_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{A} + \int_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{A} + \int_{\mathcal{S}_3} \mathbf{F} \cdot d\mathbf{A},$$

The normal to \mathcal{S}_1 and \mathcal{S}_2 is $\hat{\mathbf{k}}$ and $-\hat{\mathbf{k}}$ respectively and the area element in polar coordinates is given by $dA = r d\phi dr$ where θ is the polar angle from the positive x -axis.

For \mathcal{S}_3 the vector $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = \mathbf{r}$ points radial out from the z -axis and therefore is normal to \mathcal{S}_3 . A unit normal $\hat{\mathbf{n}}$ is constructed by dividing by a

$$\hat{\mathbf{n}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{a} = \frac{\mathbf{r}}{a}.$$

The area element along \mathcal{S}_3 is $dA = ad\theta dz$ expressed in cylindrical coordinates. Thus

$$d\mathbf{A} = \hat{\mathbf{n}} dA = \pm \hat{\mathbf{k}} r dr d\theta,$$

for \mathcal{S}_1 and \mathcal{S}_2 . While

$$d\mathbf{A} = \hat{\mathbf{n}}dA = \frac{\mathbf{r}}{a}ad\theta dz = \mathbf{r}d\theta dz$$

for \mathcal{S}_3 .

3.1 Example

Let's find the flux of the vector field

$$\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = \mathbf{r}$$

through a cylinder as described above with radius a and height L . This field points radially out from the z-axis and is therefore parallel with $d\mathbf{A}$ along \mathcal{S}_3 . We get

$$\int_{\mathcal{S}_3} \mathbf{F} \cdot d\mathbf{A} = \int_0^L \int_0^{2\pi} (x^2 + y^2) d\theta dz = \int_0^L \int_0^{2\pi} a^2 d\theta dz = 2\pi a^2 L$$

since $x^2 + y^2 = a^2$ along \mathcal{S}_3 . Another approach would be to directly observe that along this surface $\mathbf{F} \cdot d\mathbf{A} = |\mathbf{F}|dA = adA$ such that

$$\int_{\mathcal{S}_3} \mathbf{F} \cdot d\mathbf{A} = a \int_{\mathcal{S}_3} dA = 2\pi a^2 L,$$

since $\int_{\mathcal{S}_3} dA$ is just the area of \mathcal{S}_3 . For the top and bottom surfaces \mathcal{S}_1 and \mathcal{S}_2 the unit normals $\pm\hat{\mathbf{k}}$ are perpendicular to \mathbf{F} so that $\mathbf{F} \cdot d\mathbf{A} = 0$ and no flux is going through them.

As the above examples shows one can find the area vector to a surface and evaluate the flux integral using some geometry. If we want to find the flux through more general surfaces we'll need another approach. In the study of electromagnetism however, we'll mainly be concerned with flux through *closed* surfaces, and in this case there turns out to be an easier way to compute the flux than evaluating this integral directly through what is known as the *Divergence Theorem*.

Divergence of a vector field

The divergence of a vector field is a measure of the 'expansion' or 'contraction' of the field. It is also called *flux density* and is related to flux through it's definition

$$\text{Div } \mathbf{F} = \lim_{\mathcal{V} \rightarrow 0} \oint_{\mathcal{S}} \frac{\mathbf{F} \cdot d\mathbf{A}}{\mathcal{V}}, \quad (4)$$

where \mathcal{V} is the volume enclosed by the surface \mathcal{S} . I.e it's the flux out through a very tiny closed surface, shrunk until it becomes a point, divided by the volume enclosed by that surface. From this definition it follows that, just like in the case for flux, if more goes into a point \mathbf{r} than what goes out, the divergence will be negative and vice versa. These are points where vectors get shorter or longer than neighboring vectors. We say that points of negative and positive divergence are 'sinks' and 'sources' of the vector field.

4 Computation of divergence

If we were to use the definition directly we would first have to do a flux integral and then take a limit, but luckily it turns out that there is an easier way to compute the divergence. Indeed it can be shown that

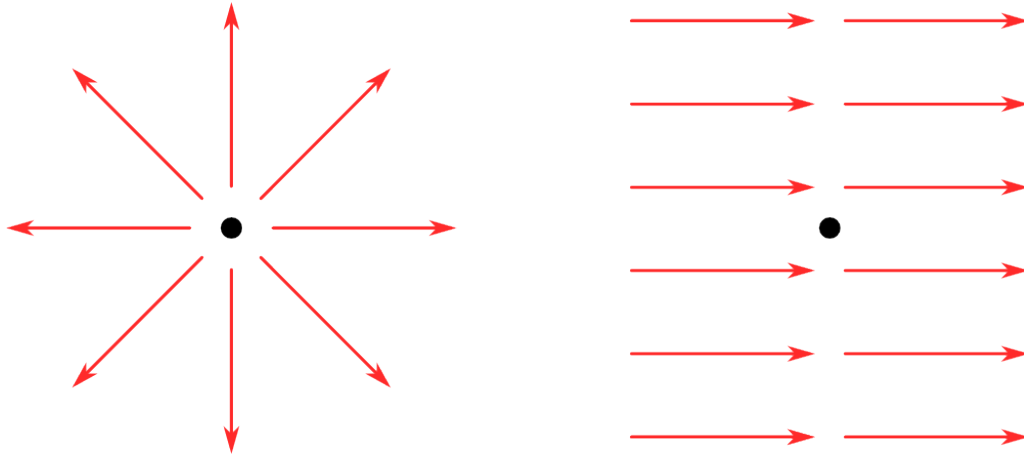


Figure 5: An example of a point of positive divergence.

$$\text{Div } \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}, \quad (5)$$

so that we can compute the divergence by taking derivatives. This is a dot-product between the del-operator and the field \mathbf{F} so we write $\text{Div } \mathbf{F} = \nabla \cdot \mathbf{F}$.

5 The Divergence Theorem

From the definition of divergence (equation 4) it should not be surprising that there is a connection between the flux through a closed surface and the divergence of the points enclosed by it. From our discussion above we found that the divergence was a measure of the sources and sinks of \mathbf{F} and the flux was a measure of how much of \mathbf{F} was going out through a surface. Now, if we consider a volume \mathcal{V} enclosed by a surface \mathcal{S} and sum up the contributions to the flow \mathbf{F} from the sources and sinks inside \mathcal{V} , it shouldn't be so far fetched to imagine that what remains of the flow \mathbf{F} must go out through the surface \mathcal{S} . This is the essence of the divergence theorem. Mathematically it states as

$$\oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{A} = \int_{\mathcal{V}} \nabla \cdot \mathbf{F} dV. \quad (6)$$

5.1 Example

Earlier in example 2.1 we computed the flux of $\mathbf{F} = cz\hat{\mathbf{k}}$ through a sphere with radius a centered at the origin. This is a closed surface so we might as well use the divergence theorem. The divergence of \mathbf{F} is

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x} \hat{\mathbf{i}} + \frac{\partial}{\partial y} \hat{\mathbf{j}} + \frac{\partial}{\partial z} \hat{\mathbf{k}} \right) \cdot cz\hat{\mathbf{k}} = c,$$

such that the flux through the sphere is

$$\oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{A} = \int_{\mathcal{V}} \nabla \cdot \mathbf{F} dV = \int_{\mathcal{V}} c dV = c \int_{\mathcal{V}} dV = \frac{4\pi a^3}{3} c.$$

where the integral $\int_{\mathcal{V}} dV$ is just the volume of the sphere.

6 Electromagnetism and Divergence

In electromagnetism we will become familiar with the four famous *Maxwell's equations* which together with the Lorentz force law beautifully describe all electromagnetic phenomenon. Here the concept of flux and divergence will become important as we study electric and magnetic fields. The first of the four equations are called Gauss' law and can be written in integral form as

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{enc}}{\epsilon_0},$$

where \mathbf{E} is the electric field and Q_{enc} is the charge enclosed by the surface \mathcal{S} . This law is telling us that the flux of the electric field through any surface \mathcal{S} is proportional to the net charge enclosed by the surface. In other words, charge give rise to electric fields, but they also terminate them. If the net charge enclosed is zero, then the total flux through the closed surface is also zero. This law can also be written on differential form through the divergence theorem. Gauss' law on differential form,

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0},$$

due to the connection between divergence and flux, carries essentially the same content, but now we're talking about a point \mathbf{r} . Here $\rho(\mathbf{r})$ is the charge density at point \mathbf{r} . The electric field diverges at a point in space where there is a charge density and thus electric charges are the sources and sinks of electric fields.

We also have similar equations for magnetic fields. Gauss' law for magnetic fields in integral

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0$$

and differential form

$$\nabla \cdot \mathbf{B} = 0$$

states that the flux or divergence of the magnetic field through any surface or at any point is always zero. This comes from the fact that there are no 'sources' or 'sinks' for magnetic fields where the field is diverging. As we'll later come to understand, the reason for this is that there exists no magnetic 'monopoles'.